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On waves in an ionized gas

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With 7 figures in the text

Introduction

In the usual magneto-ionic theory the motions of the ions and the neutral gas have always been neglected. This does not introduce any serious error at high frequencies. New types of waves are introduced by taking into account the motions of the ions and the neutral gas, as will be shown in this paper. For vanishing or infinite compressibility magneto-hydrodynamic waves are possible below the ion gyrofrequencies. When the compressibility is finite the conditions are more complicated. Then transition waves between electromagnetic (magneto-hydrodynamic) waves and sound waves occur, as has been shown by N. Herlofson.[1]

These new types of waves have been discovered by H. Alfvén. In the treatment of magneto-hydrodynamic waves in an incompressible, conducting liquid in a magnetic field B_0 he[2] used Maxwell's equations

$$\operatorname{curl} \boldsymbol{H} = \boldsymbol{i} + \frac{\partial \boldsymbol{D}}{\partial t} \tag{1}$$

$$\operatorname{curl} \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \tag{2}$$

combined with

$$\boldsymbol{i} = \sigma (\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \text{ and} \tag{3}$$

$$\varrho \cdot \frac{d\boldsymbol{v}}{dt} = \boldsymbol{i} \times \boldsymbol{B} - \operatorname{grad} p. \tag{4}$$

Since the liquid is incompressible the equation of continuity can be written

$$\operatorname{div} \boldsymbol{v} = 0; \tag{5}$$

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[1] N. Herlofson, Magneto-hydrodynamic waves in a compressible fluid conductor, Nature 165 p. 1020 (1950).

[2] H. Alfvén, Cosmical Electrodynamics, Oxford 1950, p. 76.

A treatment of these equations shows that waves can be propagated along the magnetic lines of force and that the phase velocity, for infinite conductivity, is

$$V = \sqrt{\frac{H_0 B_0}{\varrho}} \tag{6}$$

where ϱ is the mass density of the liquid.

The displacement current is negligible compared with the conduction current. The latter is capacitive, storing energy as kinetic energy, which consequently plays the same rôle as electric field energy in the vacuum wave.

In the present paper we want to study the magneto-hydrodynamic waves in an ionized gas with infinite compressibility. Since it is much easier to see the main problems in a medium without dissipation, we will assume no other forces to act on the charged particles than those from the macroscopic electric and magnetic fields (i.e. collisions are neglected). Further we want to show the connection between the magneto-hydrodynamic waves and the usual ionospheric waves.

The treatment of the problem is as in ordinary magneto-ionic theory, with the modification introduced by the motion of the ions.

The dielectric properties of a plasma

Assume a medium of free charges filling space homogeneously. There may be electrons and a finite number of types of ions, positive as well as negative. We also assume that no forces exist between these particles but only between the particles and the outer fields. Whether a neutral constituent is present or not is immaterial since, according to our assumptions, it does not influence the phenomena.

To get the dielectric properties of this medium we must know the motion of the charged particles under the influence of the electric and magnetic fields. We assume a primary magnetic field which is homogeneous and directed along the z-axis. The strength may be B. For a charged particle with the mass m and the charge e the equation of motion takes the form

$$m\dot{\boldsymbol{v}} = e\boldsymbol{E} + e\boldsymbol{v} \times \boldsymbol{B} \tag{7}$$

where a dot represents differentiation with respect to time. In order to solve this equation for $\boldsymbol{v}$ we may differentiate (7) once with respect to time and then multiply the equation vectorially with $\boldsymbol{B}$:

$$m\ddot{\boldsymbol{v}} \times \boldsymbol{B} = e\dot{\boldsymbol{E}} \times \boldsymbol{B} + e(\dot{\boldsymbol{v}} \times \boldsymbol{B}) \times \boldsymbol{B} = e\dot{\boldsymbol{E}} \times \boldsymbol{B} + e(\dot{\boldsymbol{v}}\boldsymbol{B})\boldsymbol{B} - eB^2 \cdot \dot{\boldsymbol{v}}\,. \tag{7'}$$

The scalar product between (7) and $\boldsymbol{B}$ gives

$$m\dot{\boldsymbol{v}} \cdot \boldsymbol{B} = e\boldsymbol{E}\boldsymbol{B}\,. \tag{7''}$$

Multiply the last three equations by $m^{-1}\dfrac{\partial^2}{\partial t^2}$, $e m^{-2}$ and $e^2 m^{-3} \boldsymbol{B}$ respectively and add. As a result is obtained

$$\dddot{\boldsymbol{v}} + \left(\frac{e B}{m}\right)^2 \dot{\boldsymbol{v}} = \frac{e}{m} \ddot{\boldsymbol{E}} + \frac{e}{m} \dot{\boldsymbol{E}} \times \frac{e \boldsymbol{B}}{m} + \frac{e}{m}\left(\boldsymbol{E} \cdot \frac{e \boldsymbol{B}}{m}\right) \cdot \frac{e \boldsymbol{B}}{m}. \tag{8}$$

If $\boldsymbol{v}$ and $\boldsymbol{E}$ are harmonic in time with the frequency ω we get

$$j \omega \left(\left[\frac{e B}{m}\right]^2 - \omega^2\right) \boldsymbol{v} = -\omega^2 \frac{e}{m} \boldsymbol{E} + j \omega \frac{e \boldsymbol{E}}{m} \times \frac{e \boldsymbol{B}}{m} + \frac{e}{m}\left(\boldsymbol{E} \cdot \frac{e \boldsymbol{B}}{m}\right) \cdot \frac{e \boldsymbol{B}}{m}. \tag{9}$$

If the partial density of this type of particle is n, the corresponding partial current density will be $n e \boldsymbol{v}$. The total current will be

$$\boldsymbol{i} = \sum_{\nu} n_{\nu} \cdot e_{\nu} \cdot \boldsymbol{v}_{\nu} \tag{10}$$

where the sum is to be extended over all types of particles.

To simplify the notation we may introduce

$$\omega_{\nu} = \frac{e_{\nu} B}{m_{\nu}}, \tag{11}$$

the gyrofrequency for the ν-th type of particle. This quantity has the same sign as the charge of the particle in question. Further

$$\Omega_{\nu}^2 = \frac{n_{\nu} e_{\nu}^2}{m_{\nu} \varepsilon_0}, \tag{12}$$

which may be called the plasma frequency for the ν-th type of particle. It is a suitable measure of the partial density of this type of particle.

By writing Maxwell's first equation in the form curl $\boldsymbol{H} = \dot{\boldsymbol{D}}$, we obtain the displacement current from the equation

$$\frac{\partial \boldsymbol{D}}{\partial t} = \boldsymbol{i} + \varepsilon_0 \cdot \frac{\partial \boldsymbol{E}}{\partial t}. \tag{13}$$

The equations (9), (10), and (13) give a relation between $\boldsymbol{D}$ and $\boldsymbol{E}$

$$D_{\nu} = \sum_{\mu} \varepsilon_{\nu\mu} \cdot \varepsilon_0 \cdot E_{\mu} \quad \text{or} \quad \boldsymbol{D} = (\varepsilon)\, \varepsilon_0 \cdot \boldsymbol{E}. \tag{14}$$

$\varepsilon_{\nu\mu}$ is a tensor, the dielectric tensor. With the notations (11) and (12) it gets the coefficients

$$\left.\begin{aligned}
\varepsilon_{\mathrm{I}} &= \varepsilon_{xx} + j\,\varepsilon_{xy} = 1 - \sum \frac{\Omega_\nu^2}{\omega(\omega-\omega_\nu)} = 1 + \sum \frac{\Omega_\nu^2}{\omega_\nu} \cdot \frac{1}{\omega_\nu - \omega} + \sum \frac{\Omega_\nu^2}{\omega_\nu} \cdot \frac{1}{\omega} \\
\varepsilon_{\mathrm{II}} &= \varepsilon_{xx} - j\,\varepsilon_{xy} = 1 - \sum \frac{\Omega_\nu^2}{\omega(\omega+\omega_\nu)} = 1 + \sum \frac{\Omega_\nu^2}{\omega_\nu} \cdot \frac{1}{\omega_\nu + \omega} - \sum \frac{\Omega_\nu^2}{\omega_\nu} \cdot \frac{1}{\omega} \\
\varepsilon_{\mathrm{III}} &= \varepsilon_{zz} = 1 - \sum \frac{\Omega_\nu^2}{\omega^2} \\
\varepsilon_{xx} &= \varepsilon_{yy} \\
\varepsilon_{xy} &= -\varepsilon_{yx}.
\end{aligned}\right\} \tag{15}$$

The reason for writing the coefficients in that way will be clear later.

By using (11) and (12) we can see that $\sum \frac{\Omega_\nu^2}{\omega_\nu} = \sum \frac{n_\nu e_\nu}{\varepsilon_0 B}$, and hence this quantity vanishes for a neutral medium. Then also the last sum in ε_{I} and $\varepsilon_{\mathrm{II}}$ vanishes, and ε_{I} and $\varepsilon_{\mathrm{II}}$ will be finite at the origin of the frequency space.

For a neutral medium (15) can be rewritten as

$$\left.\begin{aligned}
\varepsilon_{\mathrm{I}} &= 1 + \sum \frac{\Omega_\nu^2}{\omega_\nu} \cdot \frac{1}{\omega_\nu - \omega} \\
\varepsilon_{\mathrm{II}} &= 1 + \sum \frac{\Omega_\nu^2}{\omega_\nu} \cdot \frac{1}{\omega_\nu + \omega} \\
\varepsilon_{\mathrm{III}} &= 1 - \sum \frac{\Omega_\nu^2}{\omega^2}.
\end{aligned}\right\} \tag{15'}$$

Waves in a plasma

For our medium Maxwell's equations take the form

$$\operatorname{curl} \boldsymbol{H} = \frac{\partial \boldsymbol{D}}{\partial t} \tag{16}$$

$$\operatorname{curl} \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}. \tag{17}$$

A plasma is diamagnetic. By decreasing the temperature the diamagnetism can be made as small as desired. Here we assume the temperature so low, that the permeability is almost unity. Then

$$\boldsymbol{B} = \mu_0 \cdot \boldsymbol{H}. \tag{18}$$

Then the three last equations give the following relation between $\boldsymbol{D}$ and $\boldsymbol{E}$

$$\operatorname{curl}\operatorname{curl} \boldsymbol{E} = -\mu_0 \frac{\partial^2 \boldsymbol{D}}{\partial t^2}. \tag{19}$$

By making use of (14) we may eliminate $\boldsymbol{D}$,

$$\text{curl curl } \boldsymbol{E} = -\frac{1}{c^2} \cdot (\varepsilon) \cdot \frac{\partial^2 \boldsymbol{E}}{\partial t^2}. \tag{20}$$

We also have used the identity $c^{-2} = \varepsilon_0 \mu_0$.

To find the properties of plane waves we may assume the direction of the wave normal to be $\boldsymbol{R}\,(|\boldsymbol{R}| = 1)$. All a.c. quantities are then proportional to

$$\exp j\,\omega \left(t - \frac{\boldsymbol{R} \cdot \boldsymbol{r}}{u}\right), \tag{21}$$

where ω, as before, is the frequency of the waves, $\boldsymbol{r}$ the radius vector, and u the phase velocity.

Our vectors are of the form $\boldsymbol{A} = \boldsymbol{A}_0 \exp j\,\omega \left(t - \frac{\boldsymbol{R}\boldsymbol{r}}{u}\right)$, where $\boldsymbol{A}_0$ is independent of space and time. The curl of this vector becomes

$$\text{curl } \boldsymbol{A} = -\boldsymbol{A}_0 \times \text{grad} \exp j\,\omega \left(t - \frac{\boldsymbol{R} \cdot \boldsymbol{r}}{u}\right) =$$

$$= \frac{j\,\omega}{u} \cdot \boldsymbol{A}_0 \times \boldsymbol{R} \exp j\,\omega \left(t - \frac{\boldsymbol{R}\boldsymbol{r}}{u}\right) = \frac{j\,\omega}{u} \boldsymbol{A} \times \boldsymbol{R}.$$

Continued derivation gives

$$\text{curl curl } \boldsymbol{A} = \frac{j\,\omega}{u} \text{ curl } (\boldsymbol{A} \times \boldsymbol{R}) = \left(\frac{\omega}{u}\right)^2 \{\boldsymbol{A} - (\boldsymbol{R}\boldsymbol{A})\,\boldsymbol{R}\}.$$

Furthermore

$$\frac{\partial^2 \boldsymbol{A}}{\partial t^2} = \boldsymbol{A}_0\,(j\,\omega)^2 \exp j\,\omega \left(t - \frac{\boldsymbol{R}\boldsymbol{r}}{u}\right) = -\omega^2 \boldsymbol{A}.$$

With the two last formulas equation (20) can be transformed to

$$\left[1 - \left(\frac{u}{c}\right)^2 (\varepsilon)\right] \boldsymbol{E} - (\boldsymbol{R}\boldsymbol{E}) \cdot \boldsymbol{R} = 0. \tag{22}$$

Without loosing anything in generality, $\boldsymbol{R}$ can be assumed to be in the yz-plane and the angle between $\boldsymbol{B}_0$ and $\boldsymbol{R}$ to be φ.

The determinant Δ of (22) is

$$\Delta = -\tfrac{1}{2}\,(\varepsilon_{\mathrm{I}} + \varepsilon_{\mathrm{II}})^2\, \varepsilon_{\mathrm{III}} \cdot \left(\frac{u}{c}\right)^2 \left[\left\{\left(\frac{u}{c}\right)^2 - \frac{1}{\varepsilon_{\mathrm{I}}}\right\} \left\{\left(\frac{u}{c}\right)^2 - \frac{1}{\varepsilon_{\mathrm{II}}}\right\} \cos^2 \varphi + \right.$$

$$\left. + \left\{\left(\frac{u}{c}\right)^2 - \frac{1}{\varepsilon_{\mathrm{III}}}\right\} \left\{\left(\frac{u}{c}\right)^2 - \tfrac{1}{2}\left(\frac{1}{\varepsilon_{\mathrm{I}}} + \frac{1}{\varepsilon_{\mathrm{II}}}\right) \sin^2 \varphi\right\}\right] \cdot \tag{22'}$$

Equation (22) has a solution if the determinant vanishes. Hence we get

$$\operatorname{tg}^2 \varphi = -\frac{\left[\left(\frac{u}{c}\right)^2 - \frac{1}{\varepsilon_{\rm I}}\right]\left[\left(\frac{u}{c}\right)^2 - \frac{1}{\varepsilon_{\rm II}}\right]}{\left[\left(\frac{u}{c}\right)^2 - \frac{1}{\varepsilon_{\rm III}}\right]\left[\left(\frac{u}{c}\right)^2 - \frac{1}{2}\left(\frac{1}{\varepsilon_{\rm I}} + \frac{1}{\varepsilon_{\rm II}}\right)\right]}. \tag{23}$$

1. *R* along the primary magnetic field

In this case the numerator in (23) must vanish. We get two solutions

$$\left.\begin{aligned} \left(\frac{u}{c}\right)_1 &= \frac{1}{\sqrt{\varepsilon_{\rm I}}} \\ \left(\frac{u}{c}\right)_2 &= \frac{1}{\sqrt{\varepsilon_{\rm II}}} \end{aligned}\right\} \tag{24}$$

where $\boldsymbol{R}$ is parallel with $\boldsymbol{B}_0$.

For these solutions, the electric vector is

$$\begin{aligned} \boldsymbol{E}_1 &\sim |1, i, 0| \\ \boldsymbol{E}_2 &\sim |1, -i, 0| \end{aligned} \tag{25}$$

which is found from (22) by using (24).

The wave which is propagated in the same direction as B_0 and has the velocity u_1 is a left-handed, circularly polarized, purely transverse, electromagnetic wave. The velocity u_2 corresponds to a right-handed circularly polarized wave.

Let us plot $\left(\frac{u}{c}\right)_1^2$ as a function of ω for both positive and negative values of the latter. Positive values of ω here mean that when we look along the wave normal, we find that the electric vector of the wave rotates counter clockwise. The corresponding wave is left-handed circularly polarized. Negative values of ω mean that the electric vector is rotating in the opposite sense. A corresponding wave is right handed circularly polarized. If we neglect for a moment the negative sign for ω and use only the magnitude, we will find the phase velocity to be equal to $\left(\frac{u}{c}\right)_2$, which is defined only for positive frequencies. We thus have got a plot which in a logical way contains all waves with the wave normal along the magnetic field. In this plot there are zeros for the gyrofrequencies for all types of particles present. The mentioned frequencies are taken with sign, positive for the positive ions and negative for the negative ions as well as for the electrons. Furthermore there is one asymptote between all consecutive zeros, except in two cases. Between the lowest positive and the lowest negative zero the phase velocity is finite and positive for a neutral medium and at the origin we get

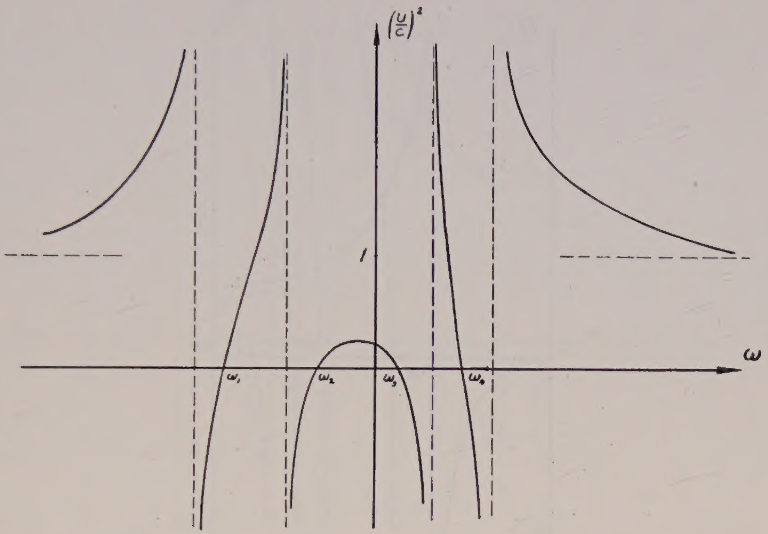


Fig. 1. Wave propagation along the magnetic lines of force in a neutral plasma. Two types of positive particles with the gyro angular frequencies ω_3 and ω_4 and two types of negative particles with the gyro angular frequencies ω_1 and ω_2.

$$\left.\begin{aligned} \frac{1}{u^2} &= \frac{1}{c^2} + \frac{\mu_0 \sum nm}{B^2} = \frac{1}{c^2} + \frac{\varrho}{BH} \\ \omega &= 0. \end{aligned}\right\} \qquad (26)$$

Here ϱ is the mass density of the ionized part of the plasma.

The other exception is above the highest positive and below the lowest negative zero. Also in this interval the phase velocity is finite, and for infinite frequency the mentioned velocity is the velocity of light.

2. *R* perpendicular to the primary magnetic field

Perpendicular to the magnetic lines of force also two waves with the same frequency are possible. One of them is a plane polarized, purely transverse wave which is independent of the magnetic field. The electric vector then is along the magnetic lines of force. For the ionized gas we may define a plasma angular frequency $\Omega = \sqrt{\sum \Omega_\nu^2}$ where Ω_ν are the partial plasma frequencies (those for the separate types of charged particles). Then we get the same form for the phase velocity as for a plasma of only electrons, (see fig. 2). Above the plasma frequency propagation is possible, below it is not. The phase velocity is always greater than the velocity of light.

The other wave has a component of the electric vector in the direction of the magnetic field. The remaining component is linearly polarized perpendicular to the magnetic field. Equation (23) shows that the phase velocity is obtained from Fig. 1 by turning the left half plane round the ordinate axis down over

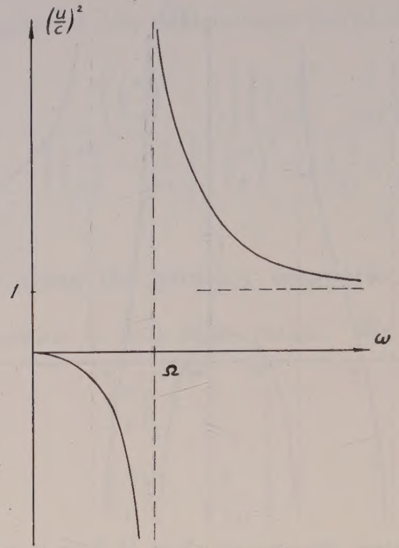


Fig. 2. Wave propagation perpendicular to the magnetic lines of force when the electric vector is parallel with the primary magnetic field. Arbitrary number of types of charged particles. Ω is the plasma angular frequency.

the right half plane and then taking the mean value of the two curves thus appearing. All the asymptotes of the two longitudinal waves will be found in the present wave.

3. Arbitrary direction of *R*

Eq. (23) is of second degree in $\left(\frac{u}{c}\right)^2$ and it always has two real roots, which may easily be seen from a plot of the equation (Fig. 4 a and b). One of the roots of the denumerator in (23) always must be halfway between the roots to the numerator. The remaining root to the denumerator $\varepsilon_{\mathrm{III}}^{-1}$, may be situated elsewhere. Two separate cases can be distinguished. This root may be nearest to $\varepsilon_{\mathrm{I}}^{-1}$ or nearest to $\varepsilon_{\mathrm{II}}^{-1}$. In the first case the wave along the magnetic field corresponding to ε_{I} continuously goes to the wave corresponding to $\varepsilon_{\mathrm{III}}$ when φ is increased to $\frac{\pi}{2}$. In the second case $\varepsilon_{\mathrm{II}}$ and $\varepsilon_{\mathrm{III}}$ correspond to each other. Generally the first one of these cases is valid in certain frequency intervals and the second in the remaining ones.

In the $\left(\frac{u}{c}\right)^2 - \varphi - \omega$-space there are two surfaces, each of which is uniquely defined if we require continuity everywhere, except at a finite number of points on the edges $\varphi = 0$ and $\varphi = \frac{\pi}{2}$. If we take the case shown in Fig. 5 we have

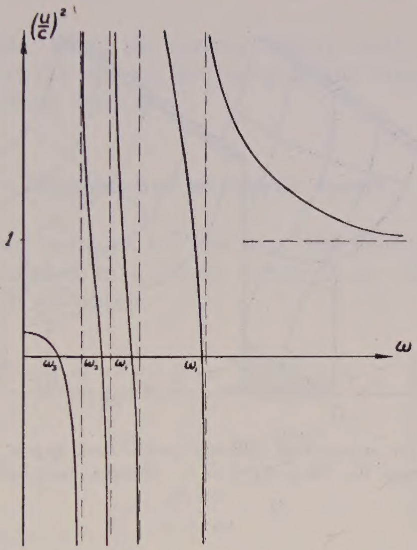
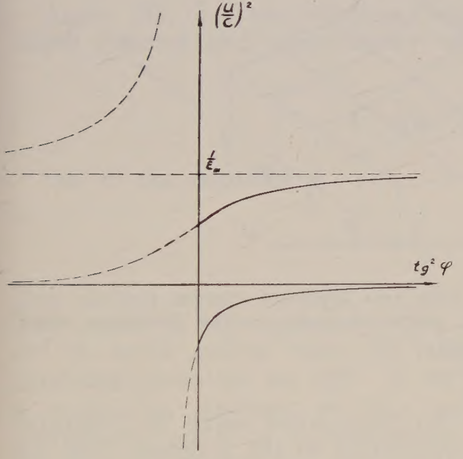
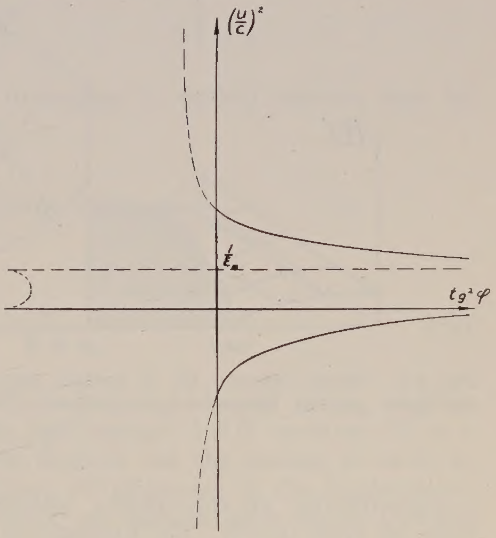


Fig. 3. Wave propagation perpendicular to the magnetic lines of force when the electric vector is perpendicular to the primary magnetic field. Plasma composition as in figure 1.



a



b

Fig. 4 a. Phase velocity as a function of the direction of the wave normal for a fixed frequency, such that there is one zero in $\left(\frac{u}{c}\right)^2$ between the horizontal asymptotes. The zero level for $\left(\frac{u}{c}\right)^2$ may fall *elsewhere* on the ordinate axis.

Fig. 4 b. Phase velocity as a function of the direction of the wave normal for a fixed frequency, such that there is no zero for $\left(\frac{u}{c}\right)^2$ between the horizontal asymptotes. The zero level for $\left(\frac{u}{c}\right)^2$ may fall elsewhere on the ordinate axis.

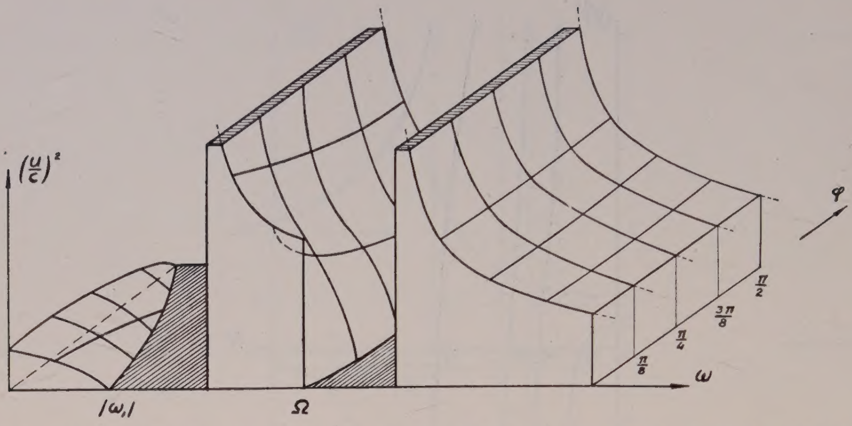


Fig. 5 a. Phase velocity for a neutral plasma with two types of charged particles with the gyro angular frequencies ω_1 and ω_2. $\Omega =$ plasma frequency. The E-surface when

$$\Omega > \frac{\omega_1 \omega_2}{\omega_1 + \omega_2}.$$

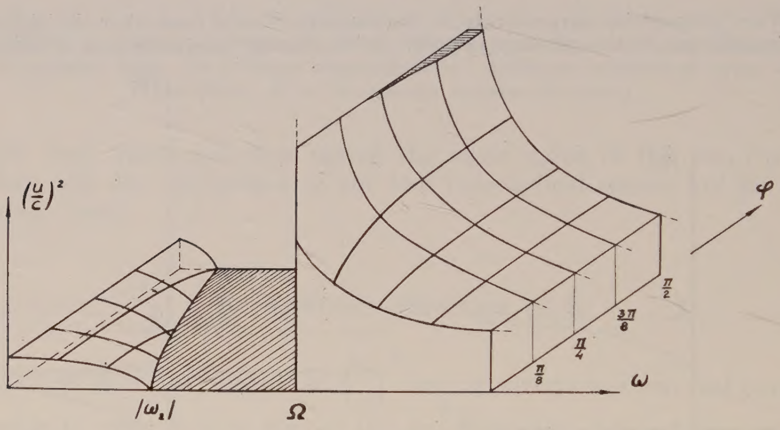


Fig. 5 b. Phase velocity for a neutral plasma with two types of charged particles with the gyro angular frequencies ω_1 and ω_2. $\Omega =$ plasma frequency. The F-surface when

$$\Omega > \frac{\omega_1 \omega_2}{\omega_1 + \omega_2}.$$

a discontinuity at the plasma frequency for propagation along the magnetic lines of force ($\varphi = 0$). For propagation perpendicular to the magnetic field we get a discontinuity in the derivate $\frac{\partial u}{\partial \omega}$ at the frequency for which the phase velocity is the same for the two possible waves. Everywhere else the surfaces in Fig. 5 a and 5 b are continuous. Let us arbitrarily call the surface giving a finite value to the refractive index when $\varphi = \frac{\pi}{2}$ and $\omega = 0$ the F-surface and the other one the E-surface.

When only two types of particles are present there are two possibilities, one at low and one at high charge density. Fig. 5 is an illustration to the latter case. In the former case, when the density is very small, the F-surface contains two infinities; the one at the plasma frequency as the lowest one. The E-surface thus contains the remaining infinity.

Magneto-hydrodynamic waves

Well below the gyrofrequencies for the ions the coefficients in the dielectric tensor (15) may be expanded in a power series in ω. Neglecting all terms except those of first order we obtain

$$\left.\begin{aligned} \varepsilon_{\mathrm{I}} &= 1 + \frac{\varrho}{\varepsilon_0 B^2} + \omega \sum \frac{n m^2}{e \varepsilon_0 B^3} + \cdots \\ \varepsilon_{\mathrm{II}} &= 1 + \frac{\varrho}{\varepsilon_0 B^2} - \omega \sum \frac{n m^2}{e \varepsilon_0 B^3} + \cdots \\ \varepsilon_{\mathrm{III}} &= 1 - \frac{\Omega^2}{\omega^2} \\ 2 \cdot \frac{\varepsilon_{\mathrm{I}} \cdot \varepsilon_{\mathrm{II}}}{\varepsilon_{\mathrm{I}} + \varepsilon_{\mathrm{II}}} &= 1 + \frac{\varrho}{\varepsilon_0 B^2} + \cdots \end{aligned}\right\} \tag{27}$$

Again (22) has two solutions, which in analogy to optical notation may be called ordinary and extraordinary waves.

1. The ordinary wave

One of the solutions is

$$\frac{1}{u_1^2} = \frac{1}{c^2} + \frac{\varrho}{B H}. \tag{28}$$

This equation shows that as long as the rest energy density of the medium ϱc^2 is much greater than the magnetic field energy, $\frac{1}{2} B H$ equation (6) is a good approximation for (28). If we for a moment use the concept of waves in a conducting medium, we may say: Since c^{-2} originates in the displacement current and $\varrho / B H$ in the conduction current, our simplification means that the displacement current has been neglected in comparison with the conduction current. Only under especial circumstances this condition is not fulfilled.

Since the phase velocity is independent of the direction of the wave normal, the wave is called the ordinary one.

Equation (22) gives the electric field strength of the wave

$$\boldsymbol{E} \approx | E_x, 0, 0 |. \tag{29}$$

Fig. 5 b shows that for propagation along the primary magnetic field this wave is right-handed circularly polarized. For low frequencies the velocity of

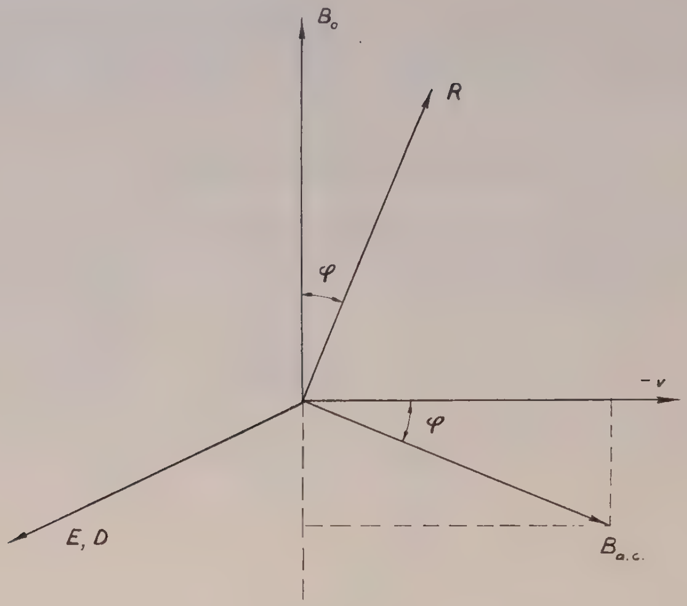


Fig. 6. Relation between the vectors characterizing the ordinary wave (magneto-hydrodynamic wave).

rotation of a linearly polarized wave is low and we may without trouble study a linearly polarized wave, which also is shown by (29). The y- and z-components vanish with the frequency and only the x-component remains finite. The electric vector is perpendicular to the magnetic field and to the wave normal (Fig. 6). Since (22) is satisfied this equation will give $\boldsymbol{D}$. It may be written

$$\boldsymbol{D} = \varepsilon_0 \cdot \left(\frac{c}{u}\right)^2 (\boldsymbol{E} - (\boldsymbol{R}\,\boldsymbol{E})\,\boldsymbol{R}). \tag{30}$$

With (29) and (28) we get

$$\boldsymbol{D} \approx \left| \varepsilon_0 \left(1 + \frac{\varrho}{\varepsilon_0 B^2}\right) E_x,\ 0,\ 0 \right|, \tag{31}$$

which shows, that $\boldsymbol{D}$ and $\boldsymbol{E}$ are parallel.

Eq. (13) gives the current

$$\boldsymbol{i} \approx \left| j\,\omega\,\frac{\varrho}{B^2} E_x,\ 0,\ 0 \right|. \tag{32}$$

Also the current is parallel to $\boldsymbol{E}$ and, as was previously mentioned, it is capacitive.

Eq. (8) gives the a.c. component of the magnetic field

$$\boldsymbol{B}_{\text{a.c.}} \approx \frac{\boldsymbol{R} \times \boldsymbol{E}}{u_2}. \tag{33}$$

As we have seen, the medium behaves as if it were isotropic, having the dielectric constant $\varepsilon = \varepsilon_{\mathrm{I}}$ (eq. 27). Equation (9) gives

$$\boldsymbol{v} = \left| 0, \; -\frac{E_x}{B}, \; 0 \right|, \tag{34}$$

which is the velocity at which no forces act on the particles.

The electric field energy $\frac{1}{2}\boldsymbol{E}\boldsymbol{D}$ is obtained from (29) and (30) as

$$\tfrac{1}{2} \cdot \boldsymbol{E}\boldsymbol{D} = \tfrac{1}{2}\varepsilon_0 E^2 + \tfrac{1}{2}\varrho v^2 = \frac{(\boldsymbol{B}_{\mathrm{a.c.}} \cdot \boldsymbol{H}_{\mathrm{a.c.}})}{2}. \tag{35}$$

Then we have also used (28) and (33). Equation (35) shows that the energy is equally divided between electric field energy and magnetic energy. The former may be divided in kinetic energy and the field energy $\frac{1}{2}\varepsilon_0 E^2$, of which the latter may be dropped when the approximate expression for the phase velocity is appropriate. For propagation along the magnetic lines of force the r.h.s. of (35) is the total a.c. component of the energy. For other directions of the wave normal, the r.h.s. of (35) gives the mean value of that energy taken over a magnetic tube of force.

For this wave $\operatorname{div} \boldsymbol{v} \neq 0$ except for a wave along the magnetic field. For a medium with finite compressibility this will cause a pressure wave and a transition between a magneto-hydrodynamic and a longitudinal sound wave is obtained. For an incompressible medium the pressure is everywhere such as to give $\operatorname{div} \boldsymbol{v} = 0$, which causes the apparent isotropy of the medium for the ordinary wave to disappear.

2. Extraordinary wave

The second solution of (22) is

$$\frac{1}{u_2^2} = \left(\frac{1}{c^2} + \frac{\varrho}{BH}\right) \frac{1}{\cos^2 \varphi}. \tag{36}$$

This shows, that if the measure the phase velocity along the magnetic lines of force we get the *constant* value

$$(c^{-2} + \varrho/\boldsymbol{B}\boldsymbol{H})^{-\frac{1}{2}}$$

which indicates that energy is propagated only along the magnetic lines of force, but the wave normal may have an arbitrary direction.

If B is not too large, (36) may be written

$$u_2 = \sqrt{\frac{BH}{\varrho}} \cdot \cos \varphi. \tag{36'}$$

This wave has the electric field in the same plane as the wave normal and the magnetic field, and (22) gives

$$\boldsymbol{E} = | 0, \; E_y, \; 0 |. \tag{37}$$

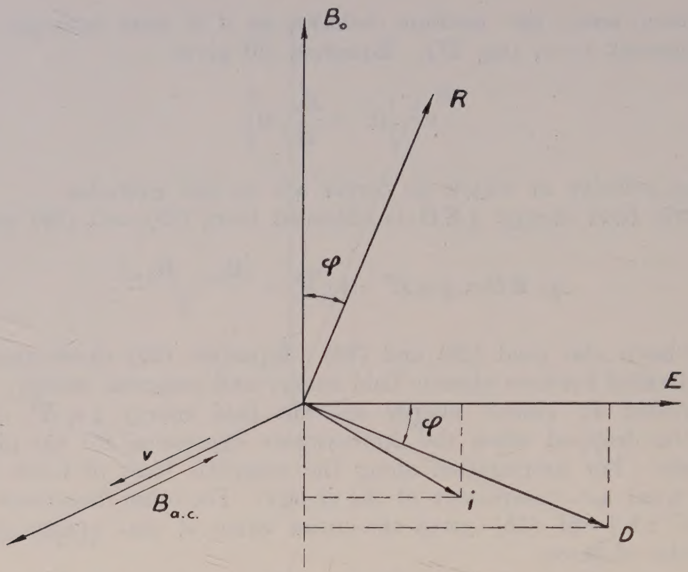


Fig. 7. Relation between the vectors characterizing the extra-ordinary wave (magneto-hydrodynamic wave).

The wave originates from the left-handed circularly polarized wave (fig. 5 a) and has degenerated to linear polarization exactly as the ordinary ray.

Equation (30) gives

$$\boldsymbol{D} = \varepsilon_0 \left(1 + \frac{\varrho}{\varepsilon_0 B^2}\right) |0,\ 1,\ -\operatorname{tg}\varphi| \cdot E_y. \tag{38}$$

D is perpendicular to the wave normal but not parallel to **E**. The reason that **D** can have a finite component along the primary magnetic field despite **E** only one of order ω^2, is the pole of second order which $\varepsilon_{\mathrm{III}}$ has for $\omega = 0$.

Equation (13) gives the current

$$\boldsymbol{i} \approx \left|0,\ j\omega\frac{\varrho}{B^2},\ -j\omega\frac{\varrho}{B^2}\left(1 + \frac{B^2\varepsilon_0}{\varrho}\right)\operatorname{tg}\varphi\right|. \tag{39}$$

Hence **B**, **R**, **E**, **D** and **i** are situated in the same plane. Both the induced magnetic field and the velocity are perpendicular to this plane. The former is obtained from (8)

$$\boldsymbol{B}_{\mathrm{a.c.}} = \frac{\boldsymbol{R} \times \boldsymbol{E}}{u_2} \tag{40}$$

and the latter from (9)

$$\boldsymbol{v} \approx \left|\frac{E}{B},\ 0,\ 0\right|, \tag{41}$$

The equations (37), (38), (40), and (41) give

$$\tfrac{1}{2}(\boldsymbol{E}\,\boldsymbol{D}) = \frac{\varepsilon_0 E^2}{2} + \frac{\varrho\, v^2}{2} = \tfrac{1}{2}(\boldsymbol{B}_{\text{a.c.}} \cdot \boldsymbol{H}_{\text{a.c.}})\,. \tag{42}$$

Here the induced magnetic field is always perpendicular to the primary one and (42) shows that we always have equipartition between the a.c. magnetic energy and the electric field energy. For the electric field energy the same conclusions as for the ordinary ray are valid.

For this wave $\operatorname{div} \boldsymbol{v} = 0$ and we then may conclude that this wave is identical with that in a liquid with infinite conductivity. Also in a medium with finite compressibility it maintains the mentioned characteristic.

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